

Order isotonicity of the metric projection onto a closed convex cone *

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Abstract

The basic tool for solving problems in metric geometry and isotonic regression is the metric projection onto closed convex cones. Isotonicity of these projections with respect to a given order relation can facilitate finding the solutions of the above problems. In the recent note [17] this problem was studied for the coordinate-wise ordering. This study was the starting point for further investigations, such as the ones presented here. The order relation in the Euclidean space endowed by a proper cone is considered and the proper cones admitting isotone metric projections with respect to this order relation are investigated.

1. Introduction

Let \mathbb{R}^m be the m -dimensional Euclidean space endowed with the standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ and the Euclidean norm $\| \cdot \|$ together with the topology this scalar product defines.

Denote by P_D the projection mapping onto a nonempty closed convex set $D \subset \mathbb{R}^m$, that is the mapping which associates to $x \in \mathbb{R}^m$ the unique nearest point $P_D x$ of x in D [25]:

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$$P_D x \in D, \text{ and } \|x - P_D x\| = \inf\{\|x - y\| : y \in D\}.$$

Given an order relation \preceq in \mathbb{R}^m , the closed convex set is called an *isotone projection set* if from $x \preceq y$, $x, y \in \mathbb{R}^m$, it follows that $P_D x \preceq P_D y$.

Due to the importance of the projection operator in applications, it is desirable to get a user friendly order relation for which the class of the isotone projection sets is as large as possible.

In this regard the interest is focused onto the widely used *vectorial ordering*, because of its natural connection to the vector-space structure of the Euclidean space \mathbb{R}^n . It is usually endowed by a cone K and is denoted by \leq_K . (See the detailed explanation for the terms which are not defined here in the next section.)

If $\preceq = \leq_K$ for some cone K , then the isotone projection set D is called *K-isotone*.

The investigations concerning the isotonicity with respect to the order relation induced by a cone of the metric projection onto a convex set go back to the paper [8] of G. Isac and A. B. Németh, where the isotone projection cones (i.e., generating pointed closed convex cones K admitting a \leq_K isotone projection onto themselves) are characterized. The same authors [9] and S. J. Bernau [2] considered the similar problem for Hilbert spaces. In these papers and in the applications in [10], [11], [19] (for the problem of solving nonlinear complementarity problems) the ordering is defined by isotone projection cones.

The next step is to get the family of closed convex sets which admit isotonic projection with respect to a given ordering. In \mathbb{R}^m with a given Cartesian reference system and the coordinate-wise order relation the problem was settled in [7], [20], [16]. If the ordering is induced by the Lorentz cone, or ice cream cone it was settled in [16]. The machinery which permits advances in this direction is to reduce the general problem to isotone projection onto subspaces. It was developed in [16] as well as in [18].

In important applications to metric geometry [4] and regression theory [1, 3, 12, 13, 21, 23, 24] the convex sets onto which the metric projection is considered are closed convex cones. The papers [15] and [5] exploited the fact that the totally ordered isotonic regression cone is an isotone projection cone too.

However, it is a very strong condition for a cone to be an isotone projection one. We would expect that considering order relations endowed by more general cones may be useful in applications. In this regard we consider in the present note the following particular case of the problem emphasized at the beginning of our introduction:

Problem: *Given a proper cone K we seek another proper cone L with the property that P_K is L -isotone.*

In our recent note [17] the family of closed convex cones admitting isotonic metric projections with respect to the coordinate-wise ordering was determined. Every isotonic regression cone belongs to this class. These results serve as justification and starting point for the other theoretical results contained in the present note.

Our investigations rely on the results in [16] as well as [18].

2. The used terminology

We aspire to be in line with the standard terminology from convex geometry. (see e.g. [22]).

The non-empty set $K \subset \mathbb{R}^m$ is called a *convex cone* if (i) $K + K \subset K$ and (ii) $tK \subset K$, $\forall t \in \mathbb{R}_+ = [0, +\infty)$. All the cones used in this paper are convex. The convex cone K is called *pointed*, if $(-K) \cap K = \{0\}$.

The convex cone K is called *generating* if $K - K = \mathbb{R}^m$.

A generating closed convex pointed cone is called *proper cone*.

For any $x, y \in \mathbb{R}^m$, by the equivalence $x \leq_K y \Leftrightarrow y - x \in K$, the convex cone K induces an *order relation* \leq_K in \mathbb{R}^m , that is, a binary relation, which is reflexive and transitive. This order relation is *translation invariant* in the sense that $x \leq_K y$ implies $x + z \leq_K y + z$ for all $z \in \mathbb{R}^m$, and *scale invariant* in the sense that $x \leq_K y$ implies $tx \leq_K ty$ for any $t \in \mathbb{R}_+$. If \leq is a translation invariant and scale invariant order relation on \mathbb{R}^m , then $\leq = \leq_K$ with $K = \{x \in \mathbb{R}^m : 0 \leq x\}$. The vector space \mathbb{R}^m endowed with the relation \leq_K is denoted by (\mathbb{R}^m, K) and is called an *ordered Euclidean vector space*. In accordance, \leq_K is called a *vectorial ordering*. If K is pointed, then \leq_K is *antisymmetric* too, that is $x \leq_K y$ and $y \leq_K x$ imply that $x = y$.

The set

$$K = \text{cone}\{x_1, \dots, x_m\} := \{t^1 x_1 + \dots + t^m x_m : t^i \in \mathbb{R}_+, i = 1, \dots, m\}$$

with x_1, \dots, x_m linearly independent vectors is called a *simplicial cone*. A simplicial cone is proper.

The *dual* of the convex cone K is the set

$$K^* := \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall x \in K\}.$$

The dual of a convex cone is a closed convex cone.

A convex cone K is called *subdual* if $K \subset K^*$ and it is called *self-dual*, if $K = K^*$. If K is self-dual, then it is proper.

Suppose that \mathbb{R}^m is endowed with a Cartesian system. Let $x, y \in \mathbb{R}^m$, $x = (x^1, \dots, x^m)$, $y = (y^1, \dots, y^m)$, where x^i, y^i are the coordinates of x and y , respectively with respect to the Cartesian system. Then, the scalar product of x and y is the sum $\langle x, y \rangle = \sum_{i=1}^m x^i y^i$.

The set

$$\mathbb{R}_+^m = \{x = (x^1, \dots, x^m) \in \mathbb{R}^m : x^i \geq 0, i = 1, \dots, m\}$$

is called the *nonnegative orthant* of the above introduced Cartesian system. It is a simplicial cone. A direct verification shows that \mathbb{R}_+^m is a self-dual cone.

Taking a Cartesian system in \mathbb{R}^m and using the above introduced notations, the *coordinatewise order* \leq in \mathbb{R}^m is defined by

$$x = (x^1, \dots, x^m) \leq y = (y^1, \dots, y^m) \Leftrightarrow x^i \leq y^i, i = 1, \dots, m.$$

By using the notion of the order relation induced by a cone, defined above, it is easy to see that $\leq = \leq_{\mathbb{R}_+^m}$.

A *hyperplane* (through $a \in \mathbb{R}^m$) is a set of form

$$H(u, a) = \{x \in \mathbb{R}^m : \langle u, x \rangle = \langle u, a \rangle\}, \quad u \neq 0. \quad (1)$$

A hyperplane $H(u, a)$ determines two *closed halfspaces* $H_-(u, a)$ and $H_+(u, a)$ of \mathbb{R}^m , defined by

$$H_-(u, a) = \{x \in \mathbb{R}^m : \langle u, x \rangle \leq \langle u, a \rangle\},$$

and

$$H_+(u, a) = \{x \in \mathbb{R}^m : \langle u, x \rangle \geq \langle u, a \rangle\}.$$

The hyperplane $H(u, 0)$ is a *supporting hyperplane* to the cone K if $K \subset H_-(u, 0)$.

The proper cone K is said *strictly convex* if the dimension $\dim(K \cap H(u, 0))$ is at most 1 for each supporting hyperplane of K . The strictly convex proper cone K is called also *smooth* if through each its boundary point $x \neq 0$ there exist exactly one supporting hyperplane to K .

The following auxiliary results are consequences of standard reasonings in convex geometry (see e. g. [22] and [25]).

Lemma 1 *Let K be a strictly convex proper cone and L be a proper cone. If $\text{int}(K) \cap L = \emptyset$, then $\dim(K \cap L) \leq 1$.*

Lemma 2 *If K is a smooth strictly convex proper cone and $H(u, 0)$ is supporting hyperplane to K through a boundary point $x \neq 0$ of K , then $P_K^{-1}(K \cap H(u, 0)) = \text{sp}\{x, u\}$ where $\text{sp} M$ stands for the linear span of the set M . Thus the set of points which projects by P_K on the ray on the boundary of K engendered by x is a two-dimensional subspace.*

An example for a smooth strictly convex proper cone is the so called Lorentz or ice cream cone:

The *Lorentz or ice cream cone* $L \subset \mathbb{R}^m \times \mathbb{R}$ is defined by

$$L = \{(x, t) \in \mathbb{R}^m \times \mathbb{R} : t \geq \|x\|\}.$$

It is a self-dual, smooth strictly convex cone. Other examples of self-dual, smooth, strictly convex cones can be found in [6].

3. Preliminary results

We will use in the following proofs the following simplified form of Moreau's decomposition theorem [14]:

Theorem 1 *Let K be a closed convex cone in \mathbb{R}^m and K^* its dual. For any x in \mathbb{R}^m we have $x = P_K x - P_{K^*}(-x)$ and $\langle P_K x, P_{K^*}(-x) \rangle = 0$. The relation $P_K x = 0$ holds if and only if $x \in -K^*$.*

One of the basic tools in our proofs is the following result which can be derived from Theorem 1, Theorem 2 and Lemma 4 in [16]:

Theorem 2 *A closed convex set $C \subset \mathbb{R}^m$ with nonempty interior is K -isotone if and only if it can be represented in the form*

$$C = \bigcap_{i \in \mathbb{N}} H_-(u_i, a_i), \quad (2)$$

where each hyperplane $H(u_i, a_i)$ is tangent to C and is K -isotone.

The next theorem follows from Theorem 1 and Lemma 4 of the above cited paper.

Theorem 3 *If C is a closed convex set, then it is K -isotone if and only if it is K^* -isotone.*

4. Isotone projection onto a proper cone

The following theorem can be considered a main result of our note, which serves also as basic tool for the next results.

Theorem 4 *Let K, L be proper cones. If K is an L -isotone projection set and $\text{int}(K^*) \cap L$ or $\text{int}(K^*) \cap L^*$ is nonempty, then K is subdual and $K \subset L \subset K^*$.*

Proof. Suppose first that $\text{int}(K^*) \cap L \neq \emptyset$ and let $v \in \text{int}(K^*) \cap L$. Then, $-v \in \text{int}(-K^*)$. Consider an arbitrary element $u \in K$ and an arbitrary positive integer n . Then, $(1/n)u - v \in -K^* \iff u - nv \in -K^*$ if n is large enough. By using $u - nv \leq_L u$, the L -isotonicity of P_K and Theorem 1, we get $0 = P_K(u - nv) \leq_L P_K(u) = u$. Thus $K \subset L$.

Hence $L^* \subset K^*$, or equivalently $K^* \cap L^* = L^*$ which, by using that L is proper (and hence L^* as well), implies $\emptyset \neq \text{int}(L^*) = \text{int}(K^* \cap L^*) = \text{int}(K^*) \cap \text{int}(L^*) \subset \text{int}(K^*) \cap L^*$. Since K is an L -isotone projection set, from Theorem 3 it follows that K is also an L^* -isotone projection set. Since $\text{int}(K^*) \cap L^* \neq \emptyset$, we can use the above reasonings to get $K \subset L^*$, which implies $L \subset K^*$. Hence, $K \subset L \subset K^*$ which also shows that K is subdual.

Next suppose that $\text{int}(K^*) \cap L^* \neq \emptyset$. Then, by using that K is an L^* -isotone projection set and the above result with L^* replacing L , we get that K is subdual and $K \subset L^* \subset K^*$, which implies $K \subset L \subset K^*$. \square

5. The case of self-dual K

We remember that the proper cone $K \subset \mathbb{R}_+^m$ is called *isotone projection cone* if it is K -isotone. A direct verification shows that \mathbb{R}_+^m is an isotone projection cone.

Corollary 1 *Let K be a self-dual cone and L a proper cone. If K is an L -isotone projection set and $\text{int}(K) \cap L$ or $\text{int}(K) \cap L^*$ is nonempty, then $K = A\mathbb{R}_+^m$, for some orthogonal matrix A . Accordingly, the only proper cones L such that \mathbb{R}_+^m is L -isotone are the orthants of the reference system.*

Proof. By Theorem 4, we get $K = L$. Thus by the main result in [8], K is a self-dual isotone projection cone, or equivalently $K = A\mathbb{R}_+^m$, for some orthogonal matrix A .

Suppose now that \mathbb{R}_+^m is L -isotone with L proper. Denote by K an orthant of the reference system, with $\text{int}(K) \cap L \neq \emptyset$. K is self-dual, hence we must have by Theorem 4 that $K \subset L$ since K is L -isotone together with \mathbb{R}_+^m (this follows from Theorem 2, K and \mathbb{R}_+^m having the same supporting hyperplanes). Now K is also self-dual, hence $L = K$ by the first part of our proof. \square

Denote by ∂ the boundary mapping of sets.

Proposition 1 *If K is a self-dual, smooth, strictly convex cone in \mathbb{R}^m with $m \geq 3$, then there is no proper cone L in \mathbb{R}^m such that K is an L -isotone projection set.*

Proof. Suppose to the contrary, that $L \subset \mathbb{R}^m$ is a proper cone such that K is a L -isotone projection set.

Let us first assume that $\text{int}(K) \cap L \neq \emptyset$. Then, by using that K is self-dual and Corollary 1, we get that $K = A\mathbb{R}_+^m$ for some orthogonal matrix A , which is absurd because K is not polyhedral.

Next, assume that

$$\text{int}(K) \cap L = \emptyset. \quad (3)$$

Since K is a L -isotone projection set, we have that $P_K(L) \subset K \cap L$. Since $\text{int}(K) \cap L = \emptyset$, we have that

$$P_K(L) \subset \partial K \cap L \subset K \cap L. \quad (4)$$

We show first that $P_K(L) \neq \{0\}$. To this end we observe that since K is L -isotone, so is $-K$ ([18] Lemma 3). The assumption $\text{int}(-K) \cap L \neq \emptyset$ would yield a contradiction as at the beginning of our proof. Hence

$$L \subset \mathbb{R}^m \setminus (\text{int}(K) \cup \text{int}(-K)).$$

Since K is self-dual, $-K = P_K^{-1}(\{0\})$ by Theorem 1. Now, L being proper, it must have points in $\mathbb{R}^m \setminus (K \cup -K)$, which confirms our claim.

We must have according to (3) and Lemma 1 that $K \cap L$ is an one-dimensional ray on the boundary of K and $P_K(L)$ is itself this ray. Now, $L \subset P_K^{-1}(P_K(L))$ is contained by Lemma 2 in a two-dimensional subspace. Hence L cannot be a proper cone. \square

6. Isotone projection onto a simplicial cone

Let $e_1, \dots, e_m \in \mathbb{R}^m$ be linearly independent and $K = \text{cone}\{e_1, \dots, e_m\}$ be a simplicial cone. Let $\mathcal{E} = \{x = (x^1, \dots, x^m)^\top \in \mathbb{R}^m : |x^i| = 1, i = 1, \dots, m\}$ and $\varepsilon \in \mathcal{E}$. Denote

$$K_\varepsilon = \text{cone}\{\varepsilon^1 e_1, \dots, \varepsilon^m e_m\}.$$

Proposition 2 *Let $K \subset \mathbb{R}^m$ be a simplicial cone and L a proper cone such that K is an L -isotone projection set. Then, there exists an $\varepsilon \in \mathcal{E}$ such that K_ε is subdual, L -isotone and $K_\varepsilon \subset L \subset K_\varepsilon^*$.*

Proof. Since $\cup_{\varepsilon \in \mathcal{E}} K_\varepsilon^* = \mathbb{R}^m$ and L is proper we have that $\text{int}(K_\varepsilon^*) \cap L \neq \emptyset$ for some $\varepsilon \in \mathcal{E}$. Since the tangent hyperplanes of K_ε coincide with the tangent hyperplanes of K it follows from Theorem 2 that K_ε is also an L -isotone projection set. Hence, the result follows from Theorem 4. \square

Denote $N = \{1, \dots, n\}$. For an index set $I \subset N$ denote $I^c = N \setminus I$ the complementary index set of I . For any vector $x \in \mathbb{R}^m$ denote by $\text{diag}(x)$ the diagonal matrix which contains x in the main diagonal such that the (i, i) -th entry of $\text{diag}(x)$ is x^i , (where any vector $y \in \mathbb{R}^m$ is written as $y = (y^1, \dots, y^m)^\top$), while its other entries are 0. A simplicial cone $K = \text{cone}\{e_1, \dots, e_m\}$ is subdual if and only if $E^\top E$ is an $m \times m$ nonnegative matrix, where $E = (e_1, \dots, e_m)$ (the matrix with columns e_i). E is called the matrix of K .

Lemma 3 *Let $K = \text{cone}\{e_1, \dots, e_m\}$ be a simplicial cone. Then, there exists a $\varepsilon \in \mathcal{E}$ such that K_ε is subdual if and only if there exists an index set $I \subset N$ such that $\langle e_i, e_j \rangle \geq 0$ for any $i, j \in I$, $\langle e_k, e_\ell \rangle \geq 0$ for any $k, \ell \in I^c$, and $\langle e_i, e_k \rangle \leq 0$ for any $i \in I$ and any $k \in I^c$.*

Proof. Let $\varepsilon \in \mathcal{E}$. Let $I = \{i \in N : \varepsilon_i = 1\}$. Then, $I^c = \{i \in N : \varepsilon_i = -1\}$. Then, the matrix of K_ε is ED , where $E = (e_1, \dots, e_m)$ and $D = \text{diag}(\varepsilon)$. Then, K_ε is subdual if and only if $DE^\top ED = (ED)^\top ED$ is nonnegative. However, $DE^\top ED$ is the matrix whose rows and columns corresponding to each index of the index set I^c are the corresponding rows and columns of $E^\top E$, respectively multiplied by -1 . Hence, $DE^\top ED$ is nonnegative if and only if $\langle e_i, e_j \rangle \geq 0$ for any $i, j \in I$, $\langle e_k, e_\ell \rangle \geq 0$ for any $k, \ell \in I^c$, and $\langle e_i, e_k \rangle \leq 0$ for any $i \in I$ and any $k \in I^c$. This follows because the (r, s) -th entry of $E^\top E$ is $\langle e_r, e_s \rangle$ for any $r, s \in N$ and therefore for any $i, j \in I$ the (i, j) -th entry of $DE^\top ED$ is $\langle e_i, e_j \rangle$, for any $k, \ell \in I^c$ the (k, ℓ) -th entry of $DE^\top ED$ is $\langle e_k, e_\ell \rangle$, and for any $i \in I$ and any $k \in I^c$ the (i, k) -th entry of $DE^\top ED$ is $-\langle e_i, e_k \rangle$. \square

Proposition 3 *Let $K = \text{cone}\{e_1, \dots, e_m\} \subset \mathbb{R}^m$ be a simplicial cone and L a proper cone such that K is an L -isotone projection set. Then, there exists an index set $I \subset N$ such that $\langle e_i, e_j \rangle \geq 0$ for any $i, j \in I$, $\langle e_k, e_\ell \rangle \geq 0$ for any $k, \ell \in I^c$, and $\langle e_i, e_k \rangle \leq 0$ for any $i \in I$ and any $k \in I^c$.*

Proof. It follows from Proposition 2 and Lemma 3. \square

Corollary 2 *Let $K = \text{cone}\{e_1, \dots, e_m\} \subset \mathbb{R}^m$ be a simplicial cone. Suppose that $i, j, k \in N$ are three pairwise distinct indices such that $\langle e_i, e_j \rangle < 0$, $\langle e_i, e_k \rangle < 0$ and $\langle e_j, e_k \rangle < 0$. Then there is no proper cone L such that K is an L -isotone projection set.*

Proof. Suppose that L is a proper cone such that K is an L -isotone projection set. From Proposition 3, there exists an index set $I \subset N$ such that one of i, j belong to I and another one to I^c , and such that similar statements hold for i, k and j, k , respectively. This leads to an obvious contradiction. Hence, there is no proper cone L such that K is an L -isotone projection set. \square

Proposition 4 *Let $K \subset \mathbb{R}^m$ be an isotone projection cone. Then, K_ε is a K -isotone projection set for any $\varepsilon \in E$.*

Proof. Since K is a K -isotone projection set and the tangent hyperplanes of K_ε coincide with the tangent hyperplanes of K , from Theorem 2 it follows that K_ε is also a K -isotone projection set. \square

Remark 1 *From this proposition it follows that for K an isotone projection cone each member of the family $\{K_\varepsilon : \varepsilon \in \mathcal{E}\}$ is a K -isotone simplicial cone. Obviously, $\text{int}(K_\varepsilon) \cap K = \emptyset$ whenever $\text{diag } \varepsilon$ is not the identity matrix. Hence by Theorem 4 in this case we must also have*

$$\text{int}(K_\varepsilon^*) \cap K = \emptyset, \text{ and } \text{int}(K_\varepsilon^*) \cap K^* = \emptyset.$$

7. The case of \mathbb{R}_+^m -isotone projection cones

To show that in contrast with Corollary 2 there exists a large class of cones which can be \mathbb{R}_+^m -isotone projection cones or more general polyhedral cones for which there are order relations with respect to which they admit isotone projections, we cite Theorem 3 in [17] (see [17] for the definition of a facet):

Theorem 5 *If K is a generating closed convex cone in \mathbb{R}^m , then it is \mathbb{R}_+^m -isotone, if and only if it is a polyhedral cone of the form*

$$K = \bigcap_{k < l} (H_-(a_{kl1}, 0) \cap H_-(a_{kl2}, 0)), \quad k, l \in \{1, \dots, m\} \quad (5)$$

where a_{kli} are nonzero vectors with $a_{kli}^k a_{kli}^l \leq 0$ and $a_{kli}^j = 0$ for $j \notin \{k, l\}$, $i = 1, 2$. Hence K possesses at most $m(m-1)$ facets. There exists a cone K of the above form with exactly $m(m-1)$ facets.

We remark that in this theorem the cone K may be a proper or only a closed and convex generating cone.

Remark 2 *As a family of simplicial subcones contained in \mathbb{R}_+^m which are \mathbb{R}_+^m -isotone we mention the family of the so called isotonic regression cones, among which the single cone which is itself an isotone projection cone too is the monotone nonnegative cone (see Corollary 1 and 2 in [17] and the definitions therein).*

Corollary 3 *If K is an \mathbb{R}_+^m -isotone proper cone, then exactly one of the alternatives*

1. $K \subset \mathbb{R}_+^m$,
2. $\text{int}(K^*) \cap \mathbb{R}_+^m = \emptyset$

holds.

Proof. If item 1 holds, then

$$\mathbb{R}_+^m = (\mathbb{R}_+^m)^* \subset K^*,$$

and hence item 2 does not hold. If item 2 does not hold, then by Theorem 4 item 1 holds. \square

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